LINEAR RESIDUALS AND GALLAI-SIMPLICIAL COMPLEXES

IMRAN ANWAR[1], ZUNAIRA KOSAR[1], SHAHEEN NAZIR[2], KHURRAM SHABBIR[3]

Abstract. In this paper, we give a new algebraic criterion for the *shellability* of (non-pure) simplicial complex Δ over $[n]$, shellable in the sense of Björner and Wachs [4]. We show that the spanning simplicial complex of doubly unicyclic graph is non-pure shellable. Moreover, we introduce the concept of Gallai-simplicial complex $\Delta_\Gamma(G)$ of a finite simple graph G. We applied the obtained criterion to discuss the shellability of Gallai simplicial complexes associated to various classes of graphs.

Key words : *shellable simplicial complex, face ring of a simplicial complex, facet ideal, pure square-free monomial ideal.*
2000 Mathematics Subject Classification: Primary 13P10, Secondary 13H10, 13F20, 13C14.

1. Introduction

The shellability of a simplicial complex is a well known combinatorial property that carries strong algebraic interpretations for instance see [11]. Algebraic criterion for the shellability of a simplicial complex has also been a reasonably important subject, firstly introduced by A. Dress [3]. Dress [3] showed that Δ is (non-pure) shellable in the sense of Björner and Wachs [4], if and only if the face ring $K[\Delta]$ is clean. Later on Herzog and Popescu [8] extended the concept for determining the shellability of multicomplexes. The shellability criterion for multicomplexes was further refined by Popescu [10]. Cleanness is well known to be the algebraic counterpart of shellability for simplicial complexes. Recently the first author along with Raza [2] gave an instructive algebraic criterion for the shellability of pure simplicial complexes in the context of the facet ideal theory. Eagon and Reiner [5] gave a translation of the pure shellability of a dual simplicial complex $\check{\Delta}$ on the monomial generators of the Stanley-Reisner ideal $I_{\mathcal{N}}(\Delta)$. Their algebraic translation gave birth to an important class of ideals known as ideals with linear quotients (Eagon-Reiner [5] called them as Dually shellable ideals).

In this paper, we give a new way to assess the shellability of (non- pure) simplicial complexes Δ in the sense of Björner and Wachs [4]. In Theorem 3.4, we give new algebraic criterion of shellability in terms of the monomial generators of $I_{\mathcal{F}}(\Delta)$. The aim of this paper is to find easy algebraic criterion of shellability and draw attention towards finding more algebraic properties of shellable complexes in the facet ideal

1. Abdus Salam School of Mathematical Sciences, G.C. University, Lahore, Pakistan.
2. Lahore University of Management Sciences, Pakistan.
3. Department of Mathematics, G.C. University, Lahore, Pakistan.

theory. We extended the concept of spanning simplicial complex for a class of non simple graphs *doubly uni-cyclic graphs* $DU_{n,m}$. We apply our algebraic criterion to prove that $\Delta_s(DU_{n,m})$ is non-pure shellable (see Theorem 4.5). In the last section, we use the concept of Gallai graph $\Gamma(G)$ of a planar graph G to introduce Gallai-simplicial complex $\Delta_\Gamma(G)$. The buildup of Gallai-simplicial complexes from a planar graph is an abstract idea, somehow, similar to building an origami shape from a plane sheet of paper by defining a crease pattern. We use a planner graph to build a 2-dimensional simplicial complex. One may think about, investing some invariants of Gallai-simplicial complexes coming through a particular classes of graphs. In this context, we discuss the shellability properties of Gallai-simplicial complexes associated to wheel graph $(W_{n+1}))$ (with $n \geq 4$) and friendship graphs $\Delta_\Gamma(F_n)$ (with $n \geq 2$).

2. Basic Setup

A *simplicial complex* Δ on the vertex set $[n]$ is a subset of $2^{[n]}$ with the property that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. The members of Δ are called *faces* and the maximal faces under inclusion are called *facets*. If $\mathcal{F}(\Delta) = \{F_1, F_2, \ldots, F_s\}$ is the set of facets of Δ, we write Δ as

$$\Delta =< F_1, F_2, \ldots, F_s > .$$

By a *subcomplex* of Δ, we mean a simplicial complex whose facet set is a subset of $\mathcal{F}(\Delta)$. We denote the *dimension of a face* $F \in \Delta$ by $\dim(F)$ and it is defined as $\dim(F) =\mid F \mid -1$. By the dimension of a simplicial complex Δ, we mean that $\dim(\Delta) = max\{\dim(F) \mid F \text{ is a } \textit{facet} \text{ in } \Delta\}$. We say that Δ is a *pure simplicial complex* of dimension d, if all the *facets* of Δ are of dimension d.
Shellability in the case of non-pure simplicial complexes was firstly defined by Björner and Wachs [4].

Definition 2.1. A simplicial complex Δ over $[n]$ is shellable if its facets can be ordered $F_1, F_2, \ldots, F_s$ such that, for all $2 \leq j \leq s$ the subcomplex

$$\hat{\Delta}_{<F_j>} =< F_1, F_2, \ldots, F_{j-1} > \cap < F_j >$$

is a pure of dimension $\dim(F_j) - 1$.

The following definitions serve as the bridge between algebra and simplicial complexes

Definition 2.2. Let Δ be a simplicial complex over $[n]$ and $S = k[x_1, \ldots, x_n]$ be the polynomial ring over an infinite field k. Let $I_{\mathcal{N}}(\Delta)$ be the ideal of S minimally generated by square-free monomials $x_{j_1}x_{j_2}\ldots x_{j_s}$, where $\{j_1, j_2, \ldots, j_s\} \subset [n]$ is not a face of Δ. $I_{\mathcal{N}}(\Delta)$ is known as *non-face ideal* or the *Stanley- Reisner ideal* of Δ. The quotient ring $S/I_{\mathcal{N}}(\Delta)$ is called the *face ring* of Δ denoted by $k[\Delta]$.

Definition 2.3. (Faridi [6]). Let Δ be a simplicial complex over $[n]$ and $S = k[x_1, \ldots, x_n]$ be the polynomial ring over an infinite field k. Let $I_{\mathcal{F}}(\Delta) \subset S$ be the monomial ideal minimally generated by square-free monomials $m_{F_1}, \ldots, m_{F_s}$

such that $m_{F_i} = x_{i_1}x_{i_2}\dots x_{i_r}$, where $F_i = \{i_1,\dots,i_r\} \subset [n]$ is a *facet* of Δ for all $i \in \{1,\dots,s\}$. $I_{\mathcal{F}}(\Delta)$ is known as the *facet ideal* of Δ.

Here, we recall the definition of pure square-free monomial ideal from [1].

Definition 2.4. Let $I \subset S$ be a square-free monomial ideal with a minimal generating system $\{g_1,\dots,g_m\}$. We say that I is a *pure square-free monomial ideal of degree d* if and only if $\mathrm{supp}(I) = \{x_1,\dots,x_n\}$[1] and $\beta_{0j}(I) = 0$[2] for all $j \neq d$.

We conclude this section with recalling following definitions from [2].

Definition 2.5. Let I be a monomial ideal in S. We define the indeg(I) as follows

$$\mathrm{indeg}(I) = min\{j\ : \beta_{0j}(I) \neq 0\}.$$

Definition 2.6. Let $I \subset S = k[x_1,\dots,x_n]$ be a monomial ideal, we say that I has *quasi-linear quotients*, if there exists a minimal monomial system of generators $m_1, m_2, \dots, m_r$ of I such that $\mathrm{indeg}(\hat{I}_{m_i}) = 1$ for all $1 < i \leq r$, where

$$\hat{I}_{m_i} = (m_1, m_2, \dots, m_{i-1}) : (m_i).$$

3. Linear residuals and shellability

In this section, we describe some new algebraic notion for explaining algebraic criterion of (non-pure) shellability Δ in the sense of Björner and Wachs [4].

Remark 3.1. First author along with Raza [2, Theorem 3.4] had shown that Δ will be a pure shellable simplicial complex if and only if $I_{\mathcal{F}}(\Delta)$ has quasi-linear quotients. But the following example shows that facet ideal $I_{\mathcal{F}}(\Delta)$ needs not to have quasi-linear quotients in the case of non-pure shellable simplicial complexes.

Example 3.2. Let Δ be a non-pure shellable simplicial complex over [5] under the shelling $\mathcal{F}(\Delta) = \{\{1,2,3,4\},\{3,4,5\},\{2,5\}\}$. It can be seen that the facet ideal $I_{\mathcal{F}}(\Delta)$ does not have quasi-linear quotients (as $\mathrm{indeg}(\hat{I}_{m_2}) = 2$).

The following definition is essential in describing non-pure shellability criterion algebraically.

Definition 3.3. Let $I \subset S = k[x_1, x_2, \dots, x_n]$ be a monomial ideal. We say that I has **linear residuals** if there exists a minimal monomial system of generators $\{m_1, m_2, \dots, m_r\}$ of I such that $\mathrm{Res}(I_i)$ is minimally generated by linear monomials for all $1 < i \leq r$, where $\mathrm{Res}(I_i) = \{u_1, u_2, \dots, u_{i-1}\}$ such that $u_k = \frac{m_i}{GCD(m_k,m_i)}$ for all $1 \leq k \leq i-1$.

The following result is a generalization of [2, Theorem 3.4].

Theorem 3.4. Let Δ be a simplicial complex of dimension d over $[n]$. Then Δ will be *shellable* if and only if $I_{\mathcal{F}}(\Delta)$ has *linear residuals.*

[1] $\mathrm{supp}(I) = \{x_j \mid x_j \text{ divides } u, \text{ with } u \in G(I)\}$

[2] graded betti number of the ideal I

Proof. Let $\Delta =< F_1, F_2, \dots, F_s >$ be a (non-pure) simplicial complex over $[n]$ of dimension d. Firstly, we show that

$$\dim(\hat{\Delta}_{<F_i>}) = dim(F_i) - \text{indeg}(\text{Res}(I_{F_i})) \quad \text{for all} \quad 2 \leq i \leq s,$$

where $m_{F_1}, m_{F_2}, \dots, m_{F_s}$ is the minimal monomial generating system of $I_{\mathcal{F}}(\Delta)$.
By 3.3, a monomial generating system of $\text{Res}(I_{F_i})$ is given as:

$$\text{Res}(I_{F_i}) = \{u_1, u_2, \dots, u_{i-1}\}$$

with $u_k = \frac{m_{F_i}}{\text{gcd}(m_{F_i}, m_{F_k})}$ for $1 \leq k \leq i-1$. Then $x_j \mid u_k$ for some $j \in [n]$ if and only if $\{j\} \in F_i \setminus F_k$. Therefore,

$$\deg(u_k) =\mid F_i \setminus F_k \mid= \dim(F_i) - \dim(F_i \cap F_k) \, \text{for all} \;\; k < i.$$

It implies that $\text{indeg}(\text{Res}(I_{F_i})) = \min\{\dim(F_i) - \dim(F_i \cap F_k) \, \text{for all} \;\; k < i\}$. Hence, we have $\text{indeg}(\text{Res}(I_{F_i})) = \dim(F_i) - \dim(< F_1, F_2, \dots F_{i-1} > \cap < F_i >)$.
Let us consider Δ be a (non-pure) shellable simplicial complex of dimension d over $[n]$. Then by 2.1, we have for all $2 \leq j \leq s$ that the subcomplex

$$\hat{\Delta}_{<F_j>} =< F_1, F_2, \dots, F_{j-1} > \cap < F_j >$$

is pure of dimension $d-1$ for a set of facets $\{F_1, F_2, \dots, F_s\}$. Therefore from above, we have $\text{Res}(I_{F_j})$ is minimally generated by linear monomials for all $2 \leq j \leq s$ as required.
Conversely, let $I_{\mathcal{F}}(\Delta)$ has linear residuals. Also, we know from above that $\deg(\text{gcd}(m_{F_j}, m_{F_i})) = \dim(F_j \cap F_i) - 1 \leq \dim(F_i) - 1$ for all $j < i$. As So, we have $\dim(\hat{\Delta}_{<F_k>}) = \dim(F_k) - 1$ for all $1 < k \leq r$. It proves that Δ is *shellable*. □

Remark 3.5. It is worth mentioning here that a pure monomial ideal $I \subset S$ having linear residual admits quasi-linear quotients.

We give here an elementary result.

Proposition 3.6. Let $I = (m_1, m_2, \dots, m_r)$ be a monomial ideal in $S = k[x_1, x_2, \dots, x_n]$ then $\deg(\text{Res}(I_i)) \leq \deg(m_i)$ for $2 \leq i \leq r$. In particular, $\deg(\text{Res}(I_i)) = \deg(m_i)$ if and only if $\text{supp}(u_i) \bigcap \bigcup_{k=1}^{i-1} \text{supp}(u_k) = \emptyset$.

Proof. Suppose on contrary that $\deg(\text{Res}(I_i)) > \deg(m_i)$ which implies there exist $u_k \in \text{Res}(I_i)$ such that $\deg(u_k) > \deg(m_i)$. But we know by definition that $u_k = \frac{m_i}{\text{GCD}(m_k, m_i)}$ for all $k < i$ and $\deg(\text{GCD}(m_k, m_i)) \geq 0$ that leads to the contradiction. □

The following corollary gives an equivalence of the two algebraic criterions of (non-pure) shellability or one can say that it is relating two different algebraic properties.

Corollary 3.7. The face ring of a simplicial complex Δ over $[n]$ is clean if and only if $I_{\mathcal{F}}(\Delta)$ has linear residuals.

Proof. We know from [3, Theorem §4], the face ring $k[\Delta]$ is clean if and only if Δ is shellable. Therefore, result follows from Theorem 3.4. □

Moreover, following corollary explores the relation between ideals with *quasi-linear quotients* and ideals with *linear residuals*.

Corollary 3.8. If a pure square-free monomial ideal $I \subset S = k[x_1, x_2, \dots, x_n]$ has quasi-linear quotients then I will have linear residuals.

Proof. We know from [2] that a pure square-free monomial ideal I with quasi-linear quotients is the facet ideal of a pure shellable simplicial complex. Therefore, I will have linear residuals by Theorem 3.4. $\square$

Theorem 3.4 can be useful in proving the Cohen-Macaulayness of the face ring of a pure simplicial complex as follows.

Corollary 3.9. If the facet ideal $I_{\mathcal{F}}(\Delta)$ of a pure simplicial complex Δ over $[n]$ has linear residuals, then the face ring $k[\Delta]$ is Cohen-Macaulay.

Remark 3.10. The class of monomial ideals with linear residuals is important in the sense that;

$$\{\text{ linear quotients}\} \subset \{\text{quasi-linear quotients}\} \subset \{\text{linear residuals}\}.$$

4. Spanning simplicial complexes of doubly uni-cyclic graphs

In this section, we introduce the concept of spanning simplicial complexes associated to a class of finite non simple connected graphs. A spanning tree of a finite connected graph $G(V, E)$ is a subtree of G that contains every vertex of G. We represent the collection of all edge-sets of the spanning trees of G by $s(G)$, in other words;

$$s(G) = \{E(T_i) \subset E, \text{ where } T_i \text{ is a spanning tree of } G\}.$$

Definition 4.1. A *doubly uni-cyclic graph* $DU_{n,m}$ is a connected graph on n vertices containing exactly one cycle of length m (with $m \leq n$) having a double edge.

The number of edges in $DU_{n,m}$ equals to the $n+1$. In particular, if $m = n$ then $DU_{n,m}$ is the n-cyclic graph having a double edge. Here we give the definition of spanning simplicial complex for connected non simple graphs.

Definition 4.2. For a non simple finite connected graph $G(V, E)$ with $s(G) = \{E_1, E_2, \dots, E_s\}$ be the edge-set of all possible spanning trees of $G(V, E)$, we define a simplicial complex $\Delta_s(G)$ on E such that the facets of $\Delta_s(G)$ are precisely the elements of $s(G)$, we call $\Delta_s(G)$ as the *spanning simplicial complex* of $G(V, E)$. In other words;

$$\Delta_s(G) = \Big\langle E_1, E_2, \dots, E_s \Big\rangle.$$

Remark 4.3. It is important to mention that the spanning simplicial complex of a finite simple connected graph is always pure. The idea behind extending this concept for non simple finite connected graphs is to explore new classes of non-pure simplicial complexes for various algebraic investigations. One can find a spanning tree systematically by *cutting-down method*, which says that spanning tree of a given finite simple connected graph is obtained by removing one edge from each cycle appearing in the graph.

Now, we fix the edge-labeling $\{e_{11}, e_{12}, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ of $DU_{n,m}$ such that $\{e_{11}, e_{12}, e_2, \ldots, e_m\}$ is the edge-set of the only cycle in $DU_{n,m}$. In the following result, we give the characterization of $s(U_{n,m})$.

Lemma 4.4. Let $DU_{n,m}$ be the *doubly uni-cyclic graph* with the edge set $E = \{e_{11}, e_{12}, e_2, \ldots, e_n\}$. A subset $E(T_i) \subset E$ will belong to $s(DU_{n,m})$ if and only if either $T_i = E \setminus \{e_i\}$ for some $i \in \{2, \ldots, m\}$ or $T_i = E \setminus \{e_{11}, e_{12}\}$. In particular;

$$s(DU_{n,m}) = \{\hat{E}_{11,12}, \hat{E}_i, \mid \hat{E}_{11,12} = E \setminus \{e_{11}, e_{12}\}, \hat{E}_i = E \setminus \{e_i\} \text{ for all } 2 \leq i \leq m\}.$$

Proof. As $DU_{n,m}$ contains only one cycle of m vertices, so its spanning trees will be obtained by just removing any edge from $\{e_2, \ldots, e_m\}$ and the only double edge $\{e_{11}, e_{12}\}$ from the cycle of $DU_{n,m}$, which proves the result. $\square$

Our main result of this section is as follows.

Theorem 4.5. Let $\Delta_s(DU_{n,m})$ be the spanning simplicial complex of doubly uni-cyclic graph $DU_{n,m}$. Then $\Delta_s(DU_{n,m})$ is non-pure shellable.

Proof. By Theorem 3.4, it is sufficient to prove that $I_{\mathcal{F}}(\Delta_s(DU_{n,m}))$ has linear residuals in $S = k[x_{11}, x_{12}, x_2, \ldots, x_n]$. By Definition 4.2, we have

$$I_{\mathcal{F}}(\Delta_s(DU_{n,m})) = (x_{\hat{E_2}}, x_{\hat{E_3}}, \ldots, x_{\hat{E_m}}, x_{\hat{E}_{11,12}})$$

with $x_{\hat{E}_i}$ the product of all variables in S except x_i. We claim that $I_{\mathcal{F}}(\Delta_s(DU_{n,m}))$ has linear residuals with respect to the ordering given above.
Let us consider $\mathrm{Res}(I_{\mathcal{F}}(\Delta_s(DU_{n,m})))_i$ for $2 \leq i \leq m$. It is clear from 3.3 that its generating system will be $\{u_1, u_2, \ldots, u_{i-1}\}$ where $u_j = \frac{x_{\hat{E}_i}}{\gcd(x_{\hat{E}_i}, x_{\hat{E}_j})}$ for all $j < i$. Then the claim follows from the fact that the degree of monomial $\gcd(x_{\hat{E}_i}, x_{\hat{E}_j})$ is $\deg(x_{\hat{E}_i}) - 1$ for all $j < i$. Now for $\mathrm{Res}(I_{\mathcal{F}}(\Delta_s(DU_{n,m})))_m$, the fact follows from the fact that $\gcd(x_{\hat{E}_j}, x_{\hat{E}_{11,12}})$ is $\deg(x_{\hat{E}_{11,12}}) - 1$ for all $2 \leq j \leq m$. $\square$

We conclude this section with the following example.

Example 4.6. Let us consider the graph $DU(6, 4)$ given below.

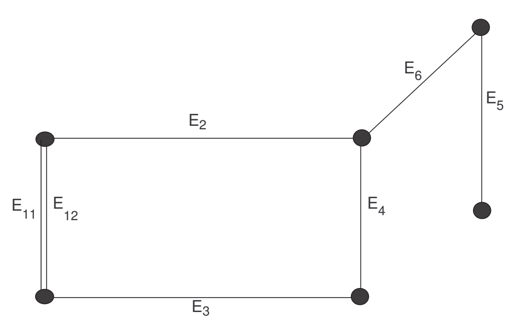


FIGURE 1. Doubly uni-cyclic graph $DU(6, 4)$

It is easy to check that $I_{\mathcal{F}}(\Delta_s(DU(6,4))) = (x_{\hat{E}_2}, x_{\hat{E}_3}, x_{\hat{E}_4}, x_{\hat{E}_{11}}, x_{\hat{E}_{12}})$ in $S = k[x_{11}, x_{12}, x_2, \ldots, x_6]$ has linear residuals with respect to the given order of minimal generators. Therefore, $\Delta_s(DU(6,4))$ is non-pure shellable simplicial complex.

5. Gallai Simplicial Complexes

Now on, G denotes a finite simple graph on the vertex set $V(G) = [n]$ and edge-set $E(G)$. The Gallai graph $\Gamma(G)$ of G is a graph whose vertex set is the edge set $E(G)$; two
distinct edges of G are adjacent in $\Gamma(G)$ if they are incident in G but do not span a triangle in G. In [9], authors discussed various combinatorial properties of Gallai and anti-Gallai graph for various classes of graphs.

The following definitions lay down the main streamline of this work.

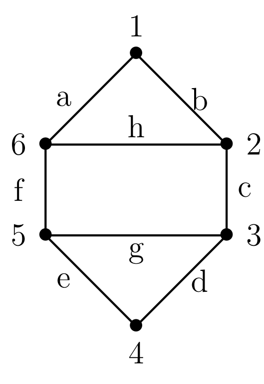
Definition 5.1. Given a graph G, its line graph $L(G)$ is a graph such that:

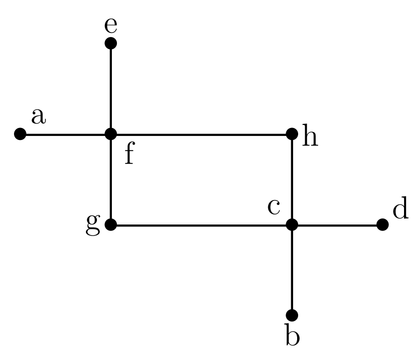
- Each vertex of $L(G)$ represents an edge of G.
- Two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G.

The following definition is a nice combinatorial buildup.

Definition 5.2. The **Gallai-graph** $\Gamma(G)$ of a graph G is the graph whose vertex set is the edge set of G; two distinct edges of G are adjacent in $\Gamma(G)$ if they are incident in G but do not span a triangle in G.

Example 5.3. Given below is a graph G and its Gallai graph $\Gamma(G)$.





G $\Gamma(G)$

The following definition is essence in the structural study of Gallai-graph $\Gamma(G)$.

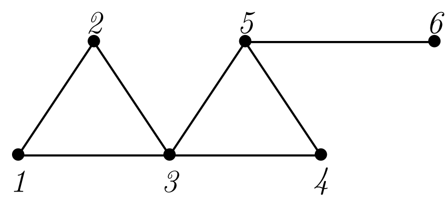
Definition 5.4. Let G be a finite simple graph with vertex set $V(G) = [n]$ and edge set $E(G) = \{e_{i,j} = \{i, j\} | i, j \in V(G)\}$.
We define the **set of Gallai-indices** $\Omega(G)$ of the graph G as the collection of subsets of $V(G)$ such that if $e_{i,j}$ and $e_{j,k}$ are adjacent in $\Gamma(G)$, then $F_{i,j,k} = \{i, j, k\} \in \Omega(G)$ or if $e_{i,j}$ is an isolated vertex in $\Gamma(G)$ then $F_{i,j} = \{i, j\} \in \Omega(G)$.

Definition 5.5. A **Gallai-simplicial complex** $\Delta_\Gamma(G)$ **of** G is a simplicial complex defined over $V(G)$ such that

$$\Delta_\Gamma(G) =< F \mid F \in \Omega(G) >,$$

where $\Omega(G)$ is the set of Gallai-indices of G.

Example 5.6. Let G be a given graph as below then its Gallai simplicial complex is as follow:
$\Delta_\Gamma(G) =< \{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 3, 5\}, \{3, 5, 6\}, \{4, 5, 6\}, \{2, 3, 4\} >$



The *wheel graph* W_{n+1} is a graph on $n + 1$ vertices with $(n \geq 4)$, formed by connecting a single vertex (called central vertex) to all vertices of an n-cycle. We fix the label of the edge-set $E(W_{n+1})$ as follows;

$$E(W_{n+1}) = \{e_{1,2}, e_{2,3}, \dots e_{n-1,n}, e_{n,1}, e_{1,n+1}, e_{2,n+1}, \dots e_{n,n+1}\}$$

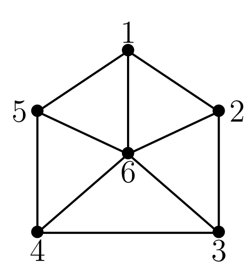
where $\{e_{1,2}, e_{2,3}, \dots e_{1,n}\}$ are edges of the n-cycle of wheel and remaining are edges inside the n-cycle.

Lemma 5.7. Let $G = W_{n+1}$ be the wheel graph on vertex set $V = [n+1]$ and edge set $E(G)$, with labelling of edges given above, then we have

$$\Omega(G) = \{F_{1,2,3}, F_{2,3,4}, \dots, F_{n-1,n,1}, F_{n,1,2}, F_{1,n+1,3}, \dots, F_{1,n+1,n-1}, F_{2,n+1,4}, \dots,$$
$$F_{2,n+1,n}, \dots, F_{n-2,n+1,n}\}.$$

Proof. From 5.4, it is clear that $F_{i,i+1,i+2} \in \Omega(G)$ because $i, i + 1, i + 2$ are vertices on n-cycle so edges $e_{i,i+1}$ and $e_{i+1,i+2}$ can not span a triangle. For internal edges we know $i, i + 1, n + 1$ makes a triangle, for all $1 \leq i \leq n$. Hence $F_{i,n+1,j} \in \Omega(G)$ for $i = 1$ we have $3 \leq j \leq n - 1$ and for $1 < i \leq n - 2$ we have $i + 2 \leq j \leq n$. $\square$

Example 5.8. For the graph given below we have its set of Gallai indices is as:
$\Omega(W_6) =< \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 1\}, \{5, 1, 2\}, \{1, 6, 3\}, \{1, 6, 4\}, \{2, 6, 4\}, \{2, 6, 5\}, \{3, 6, 5\} >$



Theorem 5.9. The face ring of Gallai-simplicial complex of wheel graph $\Delta_\Gamma(W_{n+1})$ (with $n \geq 4$), is Cohen-Macauly.

Proof. From [2], it is sufficient to show that $I_{\mathcal{F}}(\Delta_\Gamma(W_{n+1}))$ has quasi-linear quotients in $S = k[x_1, x_2, \ldots, x_{n+1}]$. By above Lemma 5.7, we have
$I_{\mathcal{F}}(\Delta_\Gamma(W_{n+1})) = (x_{F_{1,2,3}}, x_{F_{2,3,4}}, \ldots, x_{F_{n-1,n,1}}, x_{F_{n,1,2}}, x_{F_{1,n+1,3}}, \ldots, x_{F_{1,n+1,n-1}}, x_{F_{2,n+1,4}}, \ldots, x_{F_{2,n+1,n}}, \ldots, x_{F_{n-2,n+1,n}})$.
where $x_{F_{i,j,k}}$ is monomial the $x_ix_jx_k$. Now we show that $I_F(\Delta_\Gamma(G))$ has quasi-linear quotients with respect to the following order of monomials;
$m_1 = x_{F_{1,2,3}}, m_2 = x_{F_{1,2,n}}, m_3 = x_{F_{1,n-1,n}}, m_4 = x_{F_{2,3,4}}, \ldots, m_{n-1} = x_{F_{n-3,n-2,n-1}}$, $m_n = x_{F_{n-2,n-1,n}}, m^{n+1}_{1,3} = x_{F_{1,n+1,3}}, \ldots, m^{n+1}_{1,n-1} = x_{F_{1,n+1,n-1}}, m^{n+1}_{2,4} = x_{F_{2,n+1,4}}, \ldots$, $m^{n+1}_{2,n} = x_{F_{2,n+1,n}}, \ldots, m^{n+1}_{i,j} = x_{F_{i,n+1,j}}, \ldots, m^{n+1}_{n-2,n} = x_{F_{n-2,n+1,n}}$
with all m_i's for $1 \leq i \leq n$ are listed in lexicographic monomial ordering. We claim that $I_{\mathcal{F}}(\Delta_\Gamma(W_{n+1}))$ admits quasi-linear quotients for this given order. We proceed it in two steps, in the first step we show it for monomials of type m_i and in second step we show it for monomials of type $m^{n+1}_{i,j}$.
Step 1:
Let us take quotients $\widehat{I}_{m_2} = (m_1) : (m_2) = (x_3)$
$\widehat{I}_{m_3} = (m_1, m_2) : (m_3) = (x_2)$
$\widehat{I}_{m_4} = (m_1, m_2, m_3) : (m_4) = (x_3x_4, x_1)$, hence we can see all three quotients has linear term in their generating sets.
Also $\widehat{I}_{m_k} = (m_1, m_2, \ldots, m_{k-1}) : (m_k)$ has linear term due to m_{k-1} for $4 < k \leq n$. Because $m_{k-1} = x_{k-3}x_{k-2}x_{k-1}$ and $m_k = x_{k-2}x_{k-1}x_k$ are differ by one variable. Hence $\widehat{I}_{m_k} = (m_1, m_2, \ldots, m_{k-1}) : (m_k)$ has $mindeg(\hat{I_{m_k}}) = 1$ for all values of k.
Step 2:
The quotients of type $\widehat{I}_{m^{n+1}_{i,i+2}}$ for all $1 \leq i \leq n-2$, we have linear terms due to $\{m_1, m_2, \ldots, m_n\}$. As for each i, we have $x_ix_{i+1}x_{i+2} \in \{m_1, m_2, \ldots, m_n\}$. Hence assures us the presence of linear term in $\widehat{I}_{m^{n+1}_{i,i+2}}$ for all $1 \leq i \leq n-2$.

Now for $i = 1$, we have $3 < j \leq n-1$ take an arbitrary quotient $\widehat{I}_{m^{n+1}_{1,j}}$. Here we have mindeg $\widehat{I}_{m^{n+1}_{1,j}} = 1$ because of $m^{n+1}_{1,j-1}$.

Let us take an arbitrary quotient $\widehat{I}_{m^{n+1}_{i,j}}$, with $1 < i \leq n-2$ and $i+2 < j \leq n$. Therefore, we have

$$\widehat{I}_{m^{n+1}_{i,j}} = (m_1, m_2, \ldots, m_n, m^{n+1}_{1,3}, m^{n+1}_{1,4}, \ldots, m^{n+1}_{1,n-1}, m^{n+1}_{2,4}, \ldots, m^{n+1}_{i,j-1}) : (m^{n+1}_{i,j})$$

Clearly, $\widehat{I}_{m^{n+1}_{i,j}}$ has a linear term due to $m^{n+1}_{i,j-1}$. Therefore, $I_{\mathcal{F}}(\Delta_\Gamma(W_{n+1}))$ has quasi-linear quotients Hence the face ring of $\Delta_\Gamma(W_{n+1})$ is Cohen-Macaulay. $\square$

The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex. F_n is a graph on $2n+1$ vertices and $3n$ edges. Let 0 be the common vertex, label the edge-set of F_n as;

$$E(F_n) = \{e_{1,2}, e_{3,4}, \ldots, e_{2n-1,2n}, e_{1,0}, e_{2,0}, e_{3,0}, e_{4,0}, \ldots, e_{2n-1,0}, e_{2n,0}\}.$$

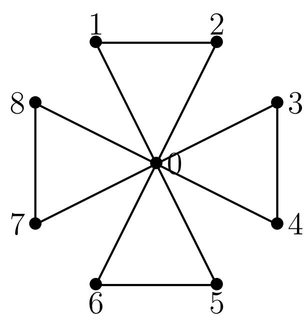
Lemma 5.10. Let $G = F_n$ be a friendship graph on the vertex set $[2n+1]$ and edge set $E(F_n)$, with labeling of edges given above we have:

$\Omega(G) = \{F_{1,2}, F_{3,4} \ldots, F_{2n-1,2n}, F_{1,0,3}, F_{1,0,4}, F_{1,0,5}, \ldots, F_{1,0,2n}, F_{2,0,3}, F_{2,0,4}, \ldots,$
$F_{2,0,2n}, \ldots, F_{2n-2,0,2n}\}$.

Proof. As by labeling of graph 0 is the common vertex. Also, $i, i+1$ together with 0 makes a triangle, whenever $1 \leq i \leq 2n-1$ and i is an odd number. Therefore, from construction of all possible triangles in F_n we have

- when i is an odd number and $i+2 \leq j \leq 2n$ then $F_{i,0,j} \in \Omega(G)$ for all $1 \leq i \leq 2n-2$.
- when i is an even number and $i+1 \leq j \leq 2n$ then $F_{i,0,j} \in \Omega(G)$ for all $1 \leq i \leq 2n-2$.
- $F_{k,k+1} \in \Omega(G)$, for $1 \leq k \leq 2n-1$ where k is an odd number.

Example 5.11. For the graph given below we have its set of Gallai indices is as:
$\Omega(F_4) = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{1,0,3\}, \{1,0,4\}, \ldots, \{1,0,8\}, \{2,0,3\}, \ldots,$
$\{2,0,8\}, \{3,0,5\}, \ldots, \{6,0,8\}\}$



Theorem 5.12. The Gallai simplicial complex of friendship graph $\Delta_\Gamma(F_n)$, with $n \geq 2$, is non-pure shellable.

Proof. By 3.4, it is enough to show that $I_{\mathcal{F}}(\Delta_\Gamma(F_n))$ has linear residuals in $S = k[x_0, x_1, x_2, x_3, \ldots, x_{2n-1}x_{2n}]$. From 5.10, we have
$I_{\mathcal{F}}(\Delta_\Gamma(G)) = (x_{F_{1,2}}, x_{F_{3,4}} \ldots, x_{F_{2n-1,2n}}, x_{F_{1,0,3}}, x_{F_{1,0,4}}, \ldots, x_{F_{1,0,2n}}, x_{F_{2,0,3}}, \ldots, x_{F_{2,0,2n}}, \ldots,$
$x_{F_{2n-2,0,2n-1}}, x_{F_{2n-2,0,2n}})$, where $x_{F_{i,0,j}}$ is the monomial $x_i x_0 x_j$ and $x_{F_{l,m}}$ is the monomial $x_l x_m$. Now we will show that $I_{\mathcal{F}}(\Delta_\Gamma(G))$ has linear residuals with respect to the following order of monomials:
$m^0_{1,3} = x_{F_{1,0,3}}, m^0_{1,4} = x_{F_{1,0,4}}, \ldots, m^0_{1,2n} = x_{F_{1,0,2n}}, m^0_{2,3} = x_{F_{2,0,3}} \ldots, m^0_{2,2n} = x_{F_{2,0,2n}}, \ldots,$
$m^0_{2n-2,2n-1} = x_{F_{2n-2,0,2n-1}}, m^0_{2n-2,2n} = x_{F_{2n-2,0,2n}}, m_{1,2} = x_{F_{1,2}}, \ldots, m_{2n-1,2n} = x_{F_{2n-1,2n}}$.
Now consider the residuals;
$\mathrm{Res}(I_{m^0_{1,4}}) = (\frac{m^0_{1,4}}{\mathrm{GCD}(m^0_{1,3}, m^0_{1,4})}) = \frac{x_1x_0x_4}{x_1x_0} = (x_4)$
$\mathrm{Res}(I_{m^0_{1,5}}) = (\frac{m^0_{1,5}}{\mathrm{GCD}(m^0_{1,3}, m^0_{1,5})}, \frac{m^0_{1,5}}{\mathrm{GCD}(m^0_{1,4}, m^0_{1,5})}) = (\frac{x_1x_0x_5}{x_1x_0}, \frac{x_1x_0x_5}{x_1x_0}) = (x_5)$
Let us take an arbitrary residual as: $\mathrm{Res}(I_{m^0_{1,j}})$, where $4 \leq j \leq 2n$, then we have

$\deg(\mathrm{GCD}(m^0_{1,j}, m^0_{1,s})) = \deg(m^0_{1,j}) - 1$ for all $3 \le j \le i-1$, so we have pure square free monomial ideal of degree 1. In particular, $\frac{m^0_{1,j}}{\mathrm{GCD}(m^0_{1,j}, m^0_{1,s})} = x_j$ for all $s < j$.
Now consider $\mathrm{Res}(I_{m^0_{i,j}})$ where $2 \le i \le 2n-2$,
Case 1.when i is even
Due to Proposition 3.6, we have $\deg(\mathrm{Res}(I_{m^0_{i,j}})) \le 2$ for $i+1 \le j \le n$, because $x_0 \in \mathrm{supp}(m^0_{i,j})$. It further implies that $\frac{m^0_{i,j}}{\mathrm{GCD}(m^0_{i_1,j_1}, m^0_{i,j})} = x_i x_j$ for all $i_1 \le i$ and $i+1 \le j_1 \le 2n$,. Moreover, there exists monomials $m^0_{i-1,j}$ (when j=i+1) and $m^0_{i,j-1}$ (when $j > i+1$) that ensure the linear residuals.
Case 2.when i is odd
Due to Proposition 3.6, we know $\deg(\mathrm{Res}(I_{m^0_{i,j}})) \le 2$ for all $i+2 \le j \le 2n$ because $x_0 \in \mathrm{supp}(m^0_{i,j})$. It further implies that $\frac{m^0_{i,j}}{\mathrm{GCD}(m^0_{i_1,j_1}, m^0_{i,j})} = x_i x_j$ for $i+2 \le j_1 \le 2n$. Moreover, there exists monomials $m^0_{i-1,j}$ (when $j = i+2$) and $m^0_{i,j-1}$ (when $j > i+2$) monomials $m^0_{i_1,j_1}$ that ensure the linear residuals.
At the end, we left with monomials of degree two so $\deg(\mathrm{Res}(I_{m_{k,k+1}})) \le 2$ for all $1 \le k \le 2n-1$ and k is an odd number. But for all k we have terms like $m^0_{k,j}$ and $m^0_{k+1,j}$ such that $\deg(\mathrm{GCD}(m_{k,k+1}, m^0_{k,j})) = \deg(m_{k,k+1}) - 1$, also $\deg(\mathrm{GCD}(m_{k,k+1}, m^0_{k+1,j})) = \deg(m_{k,k+1}) - 1$, so we have $\mathrm{Res}(I_{m_{k,k+1}})$ as a pure square free monomial ideal of degree 1. □

References

[1] Abbasi, G. Q., Ahmad, S., Anwar, I., Baig, W. A.,(2012). f-ideals of degree 2, *Algebra Colloquium*, $19(Spec1) : 921-926$.
[2] Anwar, I. Raza, Z.,(2015). Quasi-linear quotients and shellability of pure simplicial complexes, *Communications in Algebra*,43, 4698-4704.
[3] Dress, A.,(1993). A new algebraic criterion for shellability, *Beitr. Algebr. Geom.* $340(1), 45-50$.
[4] Björner, A., Wachs, M., (1997). Shellable nonpure complexes and posets. I. Trans. Amer. Math. Soc. $349(10), 3945$Ű3975.
[5] Eagon, J. A., Reiner, V.,(1998). Resolution of Stanley-Reisner Rings and Alexander Duality, *J. Pure Appl. Algebra* 130(3), 265Ű275.
[6] Faridi, S.,(2002). The facet ideal of a simplicial complex, *Manuscripta Mathematica*, 109, 159-174.
[7] Herzog, J., Hibi, T.,(2009). Monomial Algebra, Springer-Verlag New York Inc.
[8] Herzog, J., Popescu, D.,(2006). Finite Filterations of modules and shellable multicomplexes, *Manuscripta Math.* 121, 385-410.
[9] V. B. Le, Gallai graphs and anti-Gallai graphs, Discrete Math., $159, (1996), 179$Ű189
[10] Popescu, D.,(2006). Criterions for shellable multicomplexes, *An. St. Univ. Ovidius, Constanta* 14(2), 73-84.
[11] Stanley, R. P.,(1996). Combinatorics and commutative algebra, Second edition. Progress in Mathematics, 41. BirkhuserBoston , MA, 1996, ISBN: 0-8176-3836-9.

E-mail address: `imrananwar@sms.edu.pk, zunairakosar@gmail.com, shaheen.nazir@lums.edu.pk, dr.khurramshabbir@gcu.edu.pk`